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BUCKLING OF A FERROMAGNETIC CIRCULAR RING IN A RADIAL MAGNETIC FIELD

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Abstract—A possibility of the elastic buckling of a ferromagnetic circular ring in a radial magnetic field is theoretically investigated. A dipole description for the magnetization and a principle of virtual work are used to derive the equilibrium equations of the ring. The inextentionality of the buckling deformation is assumed. It is found that the ferromagnetic ring may buckle when the radial magnetic field reaches a critical value.

INTRODUCTION

The elastic buckling of ferromagnetic beams and plates in transverse magnetic fields has been studied by a number of investigators[1–8] in the recent past. One is naturally led to ask if similar phenomena are possible for a elastic ferromagnetic ring in a radial magnetic field. However, this problem is somewhat academic since radial magnetic field can only exist near the end of a bar magnet or solenoid. Hence, to the best of our knowledge, no results exist in the case of ferromagnetic rings.

The purpose of this paper is to show theoretically a possibility of the elastic buckling of a ferromagnetic circular ring in a radial magnetic field. In the present analysis, the following assumptions are employed, because of the interdisciplinary nature of the problem. The thin ring is composed of soft, linear, homogeneous ferromagnetic material and placed in a stationary radial magnetic field with no electric fields, charge distributions, or conduction currents. The magnetostrictive effect is negligible. Under these assumptions, the effect of the external magnetic field on the ring is represented by using a dipole model for magnetization. On the basis of the strain-displacement relations of Flügge's shell theory, the equations governing the behavior of rings in radial magnetic fields are derived with the aid of the principal of virtual work. Assuming the inextensional buckling deformation, the simple expression of the critical magnetic field is obtained. It is found that the critical radial magnetic field is proportional to the three-halves power of the thickness-to-radius ratio of the ring.

MAGNETIC FORCES

A ferromagnetic, thin circular ring with unit width, radius R and thickness h is considered to be set in a radial magnetic field of induction vector $\mathbf{B}_0 = B_0(R/r)\mathbf{k}$. The polar coordinate system and three unit vectors are taken as shown in Fig. 1. Let z denote distance from neutral axis of the ring, positive outward.

We assume a stationary radial magnetic field with no electric field, charge distributions, or

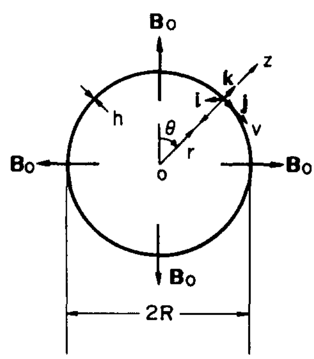

Fig. 1. Dimensions and coordinate system of the ring.

conduction currents. Under these assumptions, Maxwell's equations become

$$\nabla \times \mathbf{H} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{1}$$

where ∇ is the two-dimentional gradient operator, while $\mathbf{H}$ and $\mathbf{B}$ are the magnetizing force and the magnetic flux density, respectively. In the magnetic materials, the magnetization vector $\mathbf{M}$ is defined by $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where μ_0 is a universal constant. In a vacuum, $\mathbf{B} = \mu_0\mathbf{H}$. For soft linear ferromagnetic materials neglecting magnetostrictive effects, we assume $\mathbf{M} = \chi\mathbf{H}$, where χ is the magnetic susceptibility. Equation (1) is satisfied if

$$\mathbf{H} = \nabla\Phi, \quad \nabla^2\Phi = 0. \tag{2}$$

We assume that the edge effect of the ring on the magnetic field is negligible. Hence, the boundary conditions on the surface of the ring are

$$\mathbf{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = 0, \quad \mathbf{n} \cdot (\mathbf{B}^+ - \mathbf{B}^-) = 0 \tag{3}$$

where $\mathbf{n}$ is a unit vector normal to the surface of the ring, while the signs + and − denote the field quantities outside and inside the ring, respectively.

According to the dipole model for the magnetization, the body force and moment per unit length on the magnetized ring are given by

$$f\mathbf{k} = \int_{R-h/2}^{R+h/2} (\mathbf{M} \cdot \nabla)\mathbf{B}_0 r \, \mathrm{d}r, \quad c\mathbf{i} = \int_{R-h/2}^{R+h/2} \mathbf{M} \times \mathbf{B}_0 r \, \mathrm{d}r. \tag{4}$$

PREBUCKLING STATE

It may be expressed that the prebuckling deformation is axisymmetric and not so large. Hence, on the basis of Flügge's shell theory, the relations among stress, strain and displacement of the ring are expressed as

$$\sigma_0 = E\epsilon_0 = \frac{Ew_0}{R + z} \tag{5}$$

where E and w_0 are Young's modulus and the radial displacement, respectively. The stress resultant and moment are given by

$$N_0 = \int_{-h/2}^{h/2} \sigma_0 \, \mathrm{d}z = \frac{Eh}{R}\left(1 + \frac{h^2}{12R^2}\right) w_0, \quad M_0 = \int_{-h/2}^{h/2} \sigma_0 z \, \mathrm{d}z = -\frac{Eh^3}{12R^2} w_0. \tag{6}$$

For the axisymmetric deformation, we have $\mathbf{n} = \mathbf{k}$. Therefore, the following equation is obtained from eqns (2) and (3).

$$\mathbf{B}_0^+ = \mathbf{B}_0^- = \frac{B_0 R}{r}\mathbf{k}, \quad \mathbf{M}^- = \chi\mathbf{H}_0^- = \chi\nabla\Phi_0^- = \frac{\chi}{\mu_0(1+\chi)}\mathbf{B}_0^-. \tag{7}$$

Substitution of eqn (7) into eqn (4) yields

$$f_0 = -\frac{4\chi h B_0^2 R^2}{\mu_0(1+\chi)(4R^2 - h^2)}, \quad c_0 = 0. \tag{8}$$

The principle of virtual work yields as follows:

$$E\int_0^{2\pi} \int_{-h/2}^{h/2} \epsilon_0 \delta\epsilon_0 (R + z) \, \mathrm{d}z \, \mathrm{d}\theta = \int_0^{2\pi} f_0 \delta w_0 \, \mathrm{d}\theta. \tag{9}$$

Substituting eqns (5), (6) and (8) into eqn (9) and integrating, we obtain

$$N_0 = -\frac{4\chi h B_0^2 R_0^2}{\mu_0(1+\chi)(4R^2-h^2)}. \tag{10}$$

For the thin ring made of ferromagnetic materials ($\chi \sim 10^4$), where $1 \gg h/2R > 1/\chi$, the above equation becomes

$$N_0 \doteqdot -\frac{B_0^2 h}{\mu_0}. \tag{11}$$

BUCKLING EQUATION

In order to investigate whether the axisymmetric deformation becomes unstable, we now consider the asymmetric buckling with bifurcation of the ring. Denoting by v and w the small incremental displacement components along the neutral axis during buckling and according to Flügge's shell theory, the relations among the incremental stress, strain and displacements are expressed by

$$\sigma = E\epsilon = E(\epsilon_1+\epsilon_2), \quad \sigma_1 = E\epsilon_1, \quad \sigma_2 = E\epsilon_2$$

$$\epsilon_1 = \frac{1}{R}v_{,\theta} - \frac{z}{R(R+z)}w_{,\theta\theta} + \frac{1}{R+z}w, \quad \epsilon_2 = \frac{1}{2R^2}[(v_{,\theta}+w)^2 + (w_{,\theta}-v)^2]. \tag{12}$$

Immediately after buckling, the magnetic field quantities may be written as

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1, \quad \Phi = \Phi_0 + \Phi_1 \tag{13}$$

in which $\mathbf{B}_1$, $\mathbf{H}_1$ and Φ_1 describe the disturbed field due to the buckled ring. The equation governing the disturbed field is obtained from eqns (2) and (13) as

$$\nabla^2\Phi_1 = 0. \tag{14}$$

Since the displacements w and v are small compared with the ring thickness, we can expand the magnetic field quantities of the surface of the buckled ring in Taylor's series about $r = r_0 \equiv R + w_0 \pm h/2$[9], and obtain the magnetic flux density as follows:

$$\mathbf{B}(r_0\mathbf{k} + w\mathbf{k} + v\mathbf{j}) \doteqdot \mathbf{B}_0(r_0\mathbf{k}) + [(w\mathbf{k}+v\mathbf{j})\cdot\nabla]\mathbf{B}_0(r_0\mathbf{k}) + \mathbf{B}_1(r_0\mathbf{k}). \tag{15}$$

The unit vector normal to the surface of the buckled ring is given by

$$\mathbf{n} = \mathbf{k} - \omega\mathbf{j} \tag{16}$$

where the small rotation ω of the neutral axis of the ring is given by

$$\omega = \frac{1}{R}(w_{,\theta} - v). \tag{17}$$

Substituting eqns (15) and (16) into eqn (3) and keeping only the first terms in the displacement, we obtain

$$\Phi^+_{1,\theta} - \Phi^-_{1,\theta} = -\frac{\chi B_0}{\mu_0(1+\chi)}(w_{,\theta} - v), \quad \Phi^+_{1,r} - (1+\chi)\Phi^-_{1,r} = 0 \quad \text{at} \quad r = R + w_0 \pm \frac{h}{2}. \tag{18}$$

By using Φ_1^-, the body force and moment on the magnetized ring during buckling are given from eqn (4) as

$$f_1 = -\chi B_0 R \int_{R+w_0-h/2}^{R+w_0+h/2} \frac{1}{r}\Phi^-_{1,r}\,dr, \quad c_1 = \chi B_0 R \int_{R+w_0-h/2}^{R+w_0+h/2} \frac{1}{r}\Phi^-_{1,\theta}\,dr. \tag{19}$$

By considering the equilibrium eqn (9) at impending buckling, the variational equation of equilibrium during buckling can be given as

$$\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_0\delta\epsilon_2+\sigma_1\delta\epsilon_1)(R+z)\,\mathrm{d}z\,\mathrm{d}\theta=\int_0^{2\pi}\left[f_1\delta w+\frac{1}{R}c_1(\delta w_{,\theta}-\delta v)\right]\mathrm{d}\theta. \tag{20}$$

Assuming the inextensionality for the buckling deformation, i.e. $v_{,\theta}=-w$, we obtain the following equation from eqn (20) as

$$M_{,\theta\theta\theta}+M_{,\theta}-\left(N_0+\frac{1}{R}M_0\right)(v_{,\theta\theta\theta\theta}+2v_{,\theta\theta}+v)+Rf_{1,\theta}-c_{1,\theta\theta}-c_1=0 \tag{21}$$

where

$$M=\int_{-h/2}^{h/2}\sigma_1 z\,\mathrm{d}z=-\frac{Eh^3}{12R^2}(w_{,\theta\theta}+w). \tag{22}$$

Using eqns (6), (8) and (10), we finally obtain the buckling equation from eqn (21) as follows:

$$v_{,\theta\theta\theta\theta\theta\theta}+2v_{,\theta\theta\theta\theta}+v_{,\theta\theta}+\frac{12\chi B_0^2R^2}{\mu_0(1+\chi)Eh^2}\left(1+\frac{h^2}{12R^2}\right)^{-1}\left(1-\frac{h^2}{4R^2}\right)^{-1}(v_{,\theta\theta\theta\theta}+2v_{,\theta\theta}+v)$$
$$+\frac{12R^2}{Eh^3}(Rf_{1,\theta}-c_{1,\theta\theta}-c_1)=0. \tag{23}$$

METHOD OF SOLUTION

Considering the inextentionality for the buckling deformation, we assume the general solution of the form

$$w=A\sin n\theta,\quad v=\frac{A}{n}\cos n\theta,\quad \Phi_1=\phi(r)\sin n\theta \tag{24}$$

where A is an unknown parameter and n ($\geqq 2$) is an integer corresponding to the number of circumferential waves. Substitution of eqn (24) into eqns (14) and (18) yields

$$\phi_{,rr}+\frac{1}{r}\phi_{,r}-\frac{4}{r^2}\phi=0 \tag{25}$$

$$\phi^+-\phi^-=-\frac{(n^2-1)\chi AB_0}{n^2\mu_0(1+\chi)},\quad \phi^+_{,r}-(1+\chi)\phi^-_{,r}=0\quad\text{at}\quad r=R+w_0\pm\frac{h}{2}. \tag{26}$$

A solution of eqn (25) is given by

$$\left.\begin{aligned}&\phi^-=a_1r^2+a_1r^{-2}\quad :R+w_0-\frac{h}{2}\leqq r\leqq R+w_0+\frac{h}{2}\\&\phi^+=a_3r^2\quad :r<R+w_0-\frac{h}{2},\quad \phi^+=a_4r^{-2}\quad :r>R+w_0+\frac{h}{2}\end{aligned}\right\} \tag{27}$$

in which the arbitrary constants a_{1-4} are determined from the boundary condition (26). Hence, we obtain the following equations:

$$\Phi_1^-=-\frac{(n^2-1)AB_0}{n^2\mu_0\chi QR}[r^2+(STR^2)^2r^{-2}]\sin n\theta \tag{28}$$

where

$$S=1+\frac{w_0}{R}-\frac{h}{2R},\quad T=1+\frac{w_0}{R}+\frac{h}{2R},\quad \Gamma=2+\frac{2}{\chi},\quad Q=\Gamma[S^2-(\Gamma-1)T^2]/2. \tag{29}$$

Substitution of eqn (28) into eqn (19) yields

$$\left.\begin{aligned} f_1 &= -\frac{(n^2-1)AB_0^2}{3n\mu_0 QST}\left(\frac{h}{R}\right)^3 \sin n\theta \\ c_1 &= -\frac{2(n^2-1)AB_0^2}{n\mu_0 Q} h\left(1+\frac{w_0}{R}\right)\cos n\theta \end{aligned}\right\} . \tag{30}$$

Substituting eqns (24) and (30) into eqn (23), and considering $A \neq 0$, we obtain

$$-n^2(n^2-1)^2 + \frac{12B_0^2}{\mu_0 E}\left\{2(n^2-1)^2\left(\frac{R}{h}\right)^2\left[\Gamma ST\left(1+\frac{h^2}{12R^2}\right)\right]^{-1} - \frac{1}{Q}\left[\frac{n(n^2-1)}{3ST} + (n^2-1)^2\left(\frac{R}{h}\right)^2(S+T)\right]\right\} = 0. \tag{31}$$

When $n = 2$, we obtain the minimum value of B_0 from eqn (31). This value corresponds to the critical magnetic field B_{0c} for which the ring becomes unstable.

$$\frac{B_{0c}^2}{\mu_0 E} = \left\{6\left(\frac{R}{h}\right)^2\left[\Gamma ST\left(1+\frac{h^2}{12R^2}\right)\right]^{-1} - \frac{1}{Q}\left[\frac{2}{3ST} + 3\left(\frac{R}{h}\right)^2(S+T)\right]\right\}^{-1}. \tag{32}$$

We notice from eqn (31) that the magnetic body moment has predominant effect on the magnetoelastic buckling of the ring in radial magnetic fields. Neglecting $1/\chi$, w_0/R and $h/2R$ in comparison with unity, we obtain

$$\frac{B_{0c}^2}{\mu_0 E} \doteq \frac{1}{3}\left(\frac{h}{R}\right)^3. \tag{33}$$

This simple expression states that the critical magnetic field is independent of the magnetic susceptibility.

CONCLUSION

Theoretically, it is shown that the ferromagnetic ring may buckle when the radial magnetic field reaches a critical value. It is the magnetic body moment that dominates the magnetoelastic buckling of the ring. The critical radial magnetic field is proportional to the three-halves power of the thickness-to-radius ratio of the ring.

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